EL OPERADOR LAPLACIANO EN COORDENADAS ESFÉRICAS

Os ofrezco mi deducción completa de la expresión del operador laplaciano en coordenadas esféricas, tanto para campos escalares como para campos vectoriales. Dicha expresión aparece a menudo en física, por ejemplo, en el estudio de la propagación de ondas, o, en mecánica cuántica, para la aplicación de la ecuación de Schrödinger al átomo de hidrógeno.

Esta expresión aparece a menudo en los textos sin demostrar. En Internet pueden verse algunas demostraciones parciales; en particular, me he basado en las de Javier García (https://www.youtube.com/watch?v=Dctrfbj_Gzk) y de Adam Beatty (https://www.youtube.com/watch?v=5Uebg4AxkFo&t=48s), a los cuales agradezco su excelente labor. En mi trabajo, he completado los pasos restantes y he corregido algunos errores, casi inevitables en un cálculo tan largo. Os agradecería que me comunicarais los errores que encontréis, para poder irlos corrigiendo.

Para la notación de los ángulos en las coordenadas esféricas, puesto que, al fin y al cabo, se trata de un concepto matemático, he preferido usar la convención usual en los libros de matemáticas (al menos, en aquellos de que yo dispongo), y que me resultaba más familiar; es decir, el ángulo θ equivale al del mismo símbolo usado para las coordenadas polares en el plano XY, mientras que φ es el ángulo que forma el vector **r** con el eje Y. Quien prefiera usar la convención más habitual en los textos de física (que es la que se utiliza en las referencias citadas), no tiene más que intercambiar los signos φ y θ allí donde aparecen.

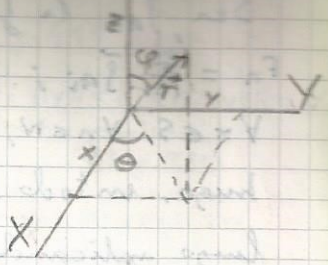
José García Illa

Mayo 2020

http://www.justaentonacion.com/jgarciailla/index.html

https://www.justaentonacion.com/

Sea un campo escalar w. Su laplaciano se define como:



$$\nabla^2 w = \nabla\cdot(\nabla w) = \frac{\partial^2 w}{\partial x} + \frac{\partial^2 w}{\partial y} + \frac{\partial^2 w}{\partial z}$$

De la figura se deduce la relación entre coordenadas rectangulares y coordenadas esféricas:

$$x = r\cos\theta\,\mathrm{sen}\,\varphi \qquad r = \sqrt{x^2+y^2+z^2}$$

$$y = r\,\mathrm{sen}\,\theta\,\mathrm{sen}\,\varphi$$

$$z = r\cos\varphi \qquad \varphi = \arccos\frac{z}{r} = \arccos\frac{z}{\sqrt{x^2+y^2+z^2}}$$

$$\theta = \arctan\frac{y}{x}$$

Aplicando la regla de la cadena:

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial w}{\partial \varphi}\frac{\partial \varphi}{\partial x} + \frac{\partial w}{\partial \theta}\frac{\partial \theta}{\partial x}\ ; \qquad \frac{\partial w}{\partial y} = \frac{\partial w}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial w}{\partial \varphi}\frac{\partial \varphi}{\partial y} + \frac{\partial w}{\partial \theta}\frac{\partial \theta}{\partial y}\ ;$$

$$\frac{\partial w}{\partial z} = \frac{\partial w}{\partial r}\frac{\partial r}{\partial z} + \frac{\partial w}{\partial \varphi}\frac{\partial \varphi}{\partial z} + \frac{\partial w}{\partial \theta}\frac{\partial \theta}{\partial z}$$

Calculemos las derivadas parciales respecto a x, y, z:

$$\frac{\partial r}{\partial x} = \frac{2x}{2\sqrt{x^2+y^2+z^2}} = \frac{x}{\sqrt{x^2+y^2+z^2}} = \frac{x}{r}, \text{ y por simetría: } \frac{\partial r}{\partial y} = \frac{y}{r}\ , \ \frac{\partial r}{\partial z} = \frac{z}{r}$$

$$\frac{\partial \varphi}{\partial x} = -\frac{1}{\sqrt{1-(z/r)^2}}\frac{\partial}{\partial x}\frac{z}{r}$$

Pero $\displaystyle\frac{\partial}{\partial x}\frac{z}{r} = z\frac{\partial}{\partial x}r^{-1} = -zr^{-2}\frac{\partial r}{\partial x} = -\frac{zx}{r^3}$

además $\sqrt{1-(z/r)^2} = \sqrt{1-\cos^2\varphi} = \mathrm{sen}\,\varphi$

Luego $\displaystyle\frac{\partial \varphi}{\partial x} = \frac{zx/r^3}{\mathrm{sen}\,\varphi} = \frac{(r\cos\varphi)(r\cos\theta\,\mathrm{sen}\,\varphi)/r^3}{\mathrm{sen}\,\varphi} = \frac{\cos\varphi\cos\theta}{r}$

$$\frac{\partial \varphi}{\partial y} = -\frac{1}{\sqrt{1-(z/r)^2}}\frac{\partial}{\partial y}\frac{z}{r}$$

Pero $\displaystyle\frac{\partial}{\partial y}\frac{z}{r} = z\frac{\partial}{\partial y}r^{-1} = -zr^{-2}\frac{\partial r}{\partial y} = -\frac{zy}{r^3}$

Luego $\displaystyle\frac{\partial \varphi}{\partial y} = \frac{zy/r^3}{\mathrm{sen}\,\varphi} = \frac{(r\cos\varphi)(r\,\mathrm{sen}\,\theta\,\mathrm{sen}\,\varphi)/r^3}{\mathrm{sen}\,\varphi} = \frac{\cos\varphi\,\mathrm{sen}\,\theta}{r}$

$$\frac{\partial \varphi}{\partial z} = -\frac{1}{\sqrt{1-(z/r)^2}}\frac{\partial}{\partial z}\frac{z}{r}$$

Pero

$$\frac{\delta}{\delta z}\frac{z}{r} = \frac{1\cdot r - z\frac{\delta r}{\delta z}}{r^2} = \frac{r - z^2/r}{r^2} = \frac{r - \frac{r^2\cos^2\varphi}{r}}{r^2} =$$

$$= \frac{r^2 - r^2\cos^2\varphi/r}{r^2} = \frac{r - r\cos^2\varphi}{r^2} = \frac{r(1-\cos^2\varphi)}{r^2} = \frac{\operatorname{sen}^2\varphi}{r}$$

Luego $\dfrac{\delta\varphi}{\delta z} = -\dfrac{\operatorname{sen}^2\varphi/r}{\operatorname{sen}\varphi} = -\dfrac{\operatorname{sen}\varphi}{r}$

$$\frac{\delta\theta}{\delta x} = \frac{1}{1+(y/x)^2}\frac{\delta}{\delta x}\frac{y}{x} = \frac{y}{1+(y/x)^2}\frac{\delta}{\delta x}\frac{1}{x} = -\frac{y/x^2}{1+(y/x)^2} =$$

$$= -\frac{y/x^2}{1+y^2/x^2} = -\frac{y/x^2}{(x^2+y^2)/x^2} = -\frac{y}{x^2+y^2} =$$

$$= -\frac{r\operatorname{sen}\theta\operatorname{sen}\varphi}{(x/\cos\theta)^2} = \frac{-r\operatorname{sen}\theta\operatorname{sen}\varphi}{r^2\cos^2\theta\operatorname{sen}^2\varphi/\cos^2\theta} = -\frac{1}{r}\frac{\operatorname{sen}\theta}{\operatorname{sen}\varphi}$$

$$\frac{\delta\theta}{\delta y} = \frac{1}{1+(y/x)^2}\frac{\delta}{\delta y}\frac{y}{x} = \frac{1/x}{1+(y/x)^2} = \frac{1/x}{(x^2+y^2)/x^2} = \frac{x}{x^2+y^2} = \frac{x}{(x/\cos\theta)^2} =$$

$$= \frac{\cos^2\theta}{x} = \frac{\cos^2\theta}{r\cos\theta\operatorname{sen}\varphi} = \frac{1}{r}\frac{\cos\theta}{\operatorname{sen}\varphi}$$

$\dfrac{\delta\theta}{\delta z} = 0$, pues θ depende de x y de y, pero no de z.

Sustituyendo estos valores en la expresión de la regla de la cadena, calcularemos las derivadas parciales primera y segunda de w respecto a x, y, z:

$$\frac{\delta w}{\delta x} = \frac{x}{r}\frac{\delta w}{\delta r} + \frac{\cos\varphi\cos\theta}{r}\frac{\delta w}{\delta\varphi} - \frac{1}{r}\frac{\operatorname{sen}\theta}{\operatorname{sen}\varphi}\frac{\delta w}{\delta\theta} =$$

$$= \cos\theta\operatorname{sen}\varphi\frac{\delta w}{\delta r} + \frac{\cos\varphi\cos\theta}{r}\frac{\delta w}{\delta\varphi} - \frac{1}{r}\frac{\operatorname{sen}\theta}{\operatorname{sen}\varphi}\frac{\delta w}{\delta\theta} = v$$

La derivada parcial de v con respecto a x será la misma expresión anterior, sustituyendo en ella w por v. Por tanto:

$$\frac{\delta^2 w}{\delta x^2} = \frac{\delta}{\delta x}\frac{\delta w}{\delta x} = \frac{\delta v}{\delta x} = \cos\theta\operatorname{sen}\varphi\frac{\delta v}{\delta r} + \frac{\cos\varphi\cos\theta}{r}\frac{\delta v}{\delta\varphi} - \frac{1}{r}\frac{\operatorname{sen}\theta}{\operatorname{sen}\varphi}\frac{\delta v}{\delta\theta} =$$

$$= \cos\theta\operatorname{sen}\varphi\frac{\delta}{\delta r}\left(\cos\theta\operatorname{sen}\varphi\frac{\delta w}{\delta r} + \frac{\cos\varphi\cos\theta}{r}\frac{\delta w}{\delta\varphi} - \frac{1}{r}\frac{\operatorname{sen}\theta}{\operatorname{sen}\varphi}\frac{\delta w}{\delta\theta}\right) +$$

$$+ \frac{\cos\varphi\cos\theta}{r}\frac{\delta}{\delta\varphi}\left(\cos\theta\operatorname{sen}\varphi\frac{\delta w}{\delta r} + \frac{\cos\varphi\cos\theta}{r}\frac{\delta w}{\delta\varphi} - \frac{1}{r}\frac{\operatorname{sen}\theta}{\operatorname{sen}\varphi}\frac{\delta w}{\delta\theta}\right) -$$

$$- \frac{1}{r}\frac{\operatorname{sen}\theta}{\operatorname{sen}\varphi}\frac{\delta}{\delta\theta}\left(\cos\theta\operatorname{sen}\varphi\frac{\delta w}{\delta r} + \frac{\cos\varphi\cos\theta}{r}\frac{\delta w}{\delta\varphi} - \frac{1}{r}\frac{\operatorname{sen}\theta}{\operatorname{sen}\varphi}\frac{\delta w}{\delta\theta}\right) =$$

$$\frac{\delta^2 w}{\delta x^2} = \cos^2\theta \operatorname{sen}^2\varphi \frac{\delta^2 w}{\delta r^2} + \cos^2\theta \operatorname{sen}\varphi \cos\varphi \left(-\frac{1}{r^2}\frac{\delta w}{\delta \varphi} + \frac{1}{r}\frac{\delta^2 w}{\delta r \delta \varphi} \right) -$$

$$- \cos\theta \operatorname{sen}\theta \left(-\frac{1}{r^2}\frac{\delta w}{\delta \theta} + \frac{1}{r}\frac{\delta^2 w}{\delta r \delta \theta} \right) +$$

$$+ \frac{\cos\varphi \cos^2\theta}{r} \left(\cos\varphi \frac{\delta w}{\delta r} + \operatorname{sen}\varphi \frac{\delta^2 w}{\delta \varphi \delta r} \right) + \frac{\cos\varphi \cos^2\theta}{r^2} \left(-\operatorname{sen}\varphi \frac{\delta w}{\delta \varphi} + \cos\varphi \frac{\delta^2 w}{\delta \varphi^2} \right) -$$

$$- \frac{\cos\varphi \cos\theta \operatorname{sen}\theta}{r^2} \left(-\frac{\cos\varphi}{\operatorname{sen}^2\varphi}\frac{\delta w}{\delta \varphi} + \frac{1}{\operatorname{sen}\varphi}\frac{\delta^2 w}{\delta \varphi \delta \theta} \right) -$$

$$- \frac{1}{r}\operatorname{sen}\theta \left(-\operatorname{sen}\theta \frac{\delta w}{\delta r} + \cos\theta \frac{\delta^2 w}{\delta \theta \delta r} \right) - \frac{\operatorname{sen}\theta \cos\varphi}{r^2 \operatorname{sen}\varphi} \left(-\operatorname{sen}\theta \frac{\delta w}{\delta \varphi} + \cos\theta \frac{\delta^2 w}{\delta \theta \delta \varphi} \right) +$$

$$+ \frac{\operatorname{sen}\theta}{\operatorname{sen}^2\varphi} \left(\cos\theta \frac{\delta w}{\delta \theta} + \operatorname{sen}\theta \frac{\delta^2 w}{\delta \theta^2} \right) =$$

$$= \operatorname{sen}^2\varphi \cos^2\theta \frac{\delta^2 w}{\delta r^2} - \frac{\operatorname{sen}\varphi \cos\varphi \cos^2\theta}{r^2}\frac{\delta w}{\delta \varphi} + \frac{\operatorname{sen}\varphi \cos\varphi \cos^2\theta}{r}\frac{\delta^2 w}{\delta r \delta \varphi} +$$

$$+ \frac{\cos\theta \operatorname{sen}\theta}{r^2}\frac{\delta w}{\delta \theta} - \frac{\cos\theta \operatorname{sen}\theta}{r}\frac{\delta^2 w}{\delta r \delta \theta} +$$

$$+ \frac{\cos^2\varphi \cos^2\theta}{r}\frac{\delta w}{\delta r} + \frac{\operatorname{sen}\varphi \cos\varphi \cos^2\theta}{r}\frac{\delta^2 w}{\delta \varphi \delta r} - \frac{\operatorname{sen}\varphi \cos\varphi \cos^2\theta}{r^2}\frac{\delta w}{\delta \varphi} +$$

$$+ \frac{\cos^2\varphi \cos^2\theta}{r^2}\frac{\delta^2 w}{\delta \varphi^2} + \frac{\cos^2\varphi \operatorname{sen}\theta \cos\theta}{r^2 \operatorname{sen}^2\varphi}\frac{\delta w}{\delta \theta} - \frac{\cos\varphi \operatorname{sen}\theta \cos\theta}{r^2 \operatorname{sen}\varphi}\frac{\delta^2 w}{\delta \varphi \delta \theta} +$$

$$+ \frac{\operatorname{sen}^2\theta}{r}\frac{\delta w}{\delta r} - \frac{\operatorname{sen}\theta \cos\theta}{r}\frac{\delta^2 w}{\delta \theta \delta r} + \frac{\operatorname{sen}^2\theta \cos\varphi}{r^2 \operatorname{sen}\varphi}\frac{\delta w}{\delta \varphi} -$$

$$- \frac{\cos\varphi \operatorname{sen}\theta \cos\theta}{r^2 \operatorname{sen}\varphi}\frac{\delta^2 w}{\delta \theta \delta \varphi} + \frac{\operatorname{sen}\theta \cos\theta}{r^2 \operatorname{sen}^2\varphi}\frac{\delta w}{\delta \theta} + \frac{\operatorname{sen}^2\theta}{r^2 \operatorname{sen}^2\varphi}\frac{\delta^2 w}{\delta \theta^2}$$

De manera similar:

$$\frac{\delta w}{\delta y} = \frac{y}{r}\frac{\delta w}{\delta r} + \frac{\cos\varphi \operatorname{sen}\theta}{r}\frac{\delta w}{\delta \varphi} + \frac{\cos\theta}{r \operatorname{sen}\varphi}\frac{\delta w}{\delta \theta} =$$

$$= \operatorname{sen}\theta \operatorname{sen}\varphi \frac{\delta w}{\delta r} + \frac{\cos\varphi \operatorname{sen}\theta}{r}\frac{\delta w}{\delta \varphi} + \frac{\cos\theta}{r \operatorname{sen}\varphi}\frac{\delta w}{\delta \theta} = h$$

$$\frac{\delta^2 w}{\delta y^2} = \frac{\delta}{\delta y}\frac{\delta w}{\delta y} = \frac{\delta h}{\delta y} = \operatorname{sen}\theta \operatorname{sen}\varphi \frac{\delta h}{\delta r} + \frac{\cos\varphi \operatorname{sen}\theta}{r}\frac{\delta h}{\delta \varphi} + \frac{\cos\theta}{r \operatorname{sen}\varphi}\frac{\delta h}{\delta \theta} =$$

$$= \operatorname{sen}\theta \operatorname{sen}\varphi \frac{\delta}{\delta r}\left(\operatorname{sen}\theta \operatorname{sen}\varphi \frac{\delta w}{\delta r} + \frac{\cos\varphi \operatorname{sen}\theta}{r}\frac{\delta w}{\delta \varphi} + \frac{\cos\theta}{r \operatorname{sen}\varphi}\frac{\delta w}{\delta \theta} \right) +$$

$$+ \frac{\cos\varphi \operatorname{sen}\theta}{r}\frac{\delta}{\delta \varphi}\left(\operatorname{sen}\theta \operatorname{sen}\varphi \frac{\delta w}{\delta r} + \frac{\cos\varphi \operatorname{sen}\theta}{r}\frac{\delta w}{\delta \varphi} + \frac{\cos\theta}{r \operatorname{sen}\varphi}\frac{\delta w}{\delta \theta} \right) +$$

$$+\frac{\cos\theta}{r\,\mathrm{sen}\varphi}\frac{\delta}{\delta\theta}\left(\mathrm{sen}\theta\,\mathrm{sen}\varphi\frac{\delta w}{\delta r}+\frac{\cos\varphi\,\mathrm{sen}\theta}{r}\frac{\delta w}{\delta\varphi}+\frac{\cos\theta}{r\,\mathrm{sen}\varphi}\frac{\delta w}{\delta\theta}\right)=$$

$$=\mathrm{sen}^2\theta\,\mathrm{sen}^2\varphi\frac{\delta^2 w}{\delta r}+\mathrm{sen}^2\theta\,\mathrm{sen}\varphi\cos\varphi\left(-\frac{1}{r^2}\frac{\delta w}{\delta\varphi}+\frac{1}{r}\frac{\delta^2 w}{\delta r\,\delta\varphi}\right)+$$

$$+\mathrm{sen}\theta\cos\theta\left(-\frac{1}{r^2}\frac{\delta w}{\delta\theta}+\frac{1}{r}\frac{\delta^2 w}{\delta r\,\delta\theta}\right)+$$

$$+\frac{\cos\varphi\,\mathrm{sen}^2\theta}{r}\left(\cos\varphi\frac{\delta w}{\delta r}+\mathrm{sen}\varphi\frac{\delta^2 w}{\delta\varphi\,\delta r}\right)+\frac{\cos\varphi\,\mathrm{sen}^2\theta}{r^2}\left(-\mathrm{sen}\varphi\frac{\delta w}{\delta\varphi}+\cos\varphi\frac{\delta^2 w}{\delta\varphi^2}\right)+$$

$$+\frac{\cos\varphi\,\mathrm{sen}\theta\cos\theta}{r^2}\left(-\frac{\cos\varphi}{\mathrm{sen}^2\varphi}\frac{\delta w}{\delta\theta}+\frac{1}{\mathrm{sen}\varphi}\frac{\delta^2 w}{\delta\varphi\,\delta\theta}\right)+$$

$$+\frac{\cos\theta}{r}\left(\cos\theta\frac{\delta w}{\delta r}+\mathrm{sen}\theta\frac{\delta^2 w}{\delta\theta\,\delta r}\right)+\frac{\cos\theta\cos\varphi}{r^2\,\mathrm{sen}\varphi}\left(\cos\theta\frac{\delta w}{\delta\varphi}+\mathrm{sen}\theta\frac{\delta^2 w}{\delta\theta\,\delta\varphi}\right)+$$

$$+\frac{\cos\theta}{r^2\,\mathrm{sen}^2\varphi}\left(-\mathrm{sen}\theta\frac{\delta w}{\delta\theta}+\cos\theta\frac{\delta^2 w}{\delta\theta^2}\right)=$$

$$=\mathrm{sen}^2\varphi\,\mathrm{sen}^2\theta\frac{\delta^2 w}{\delta r^2}-\frac{\mathrm{sen}\varphi\cos\varphi\,\mathrm{sen}^2\theta}{r^2}\frac{\delta w}{\delta\varphi}+\frac{\mathrm{sen}\varphi\cos\varphi\,\mathrm{sen}^2\theta}{r}\frac{\delta^2 w}{\delta r\,\delta\varphi}-$$

$$-\frac{\mathrm{sen}\theta\cos\theta}{r^2}\frac{\delta w}{\delta\theta}+\frac{\mathrm{sen}\theta\cos\theta}{r}\frac{\delta^2 w}{\delta r\,\delta\theta}+$$

$$+\frac{\cos^2\varphi\,\mathrm{sen}^2\theta}{r}\frac{\delta w}{\delta r}+\frac{\mathrm{sen}\varphi\cos\varphi\,\mathrm{sen}^2\theta}{r}\frac{\delta^2 w}{\delta\varphi\,\delta r}-\frac{\mathrm{sen}\varphi\cos\varphi\,\mathrm{sen}^2\theta}{r^2}\frac{\delta w}{\delta\varphi}+$$

$$+\frac{\cos^2\varphi\,\mathrm{sen}^2\theta}{r^2}\frac{\delta^2 w}{\delta\varphi^2}-\frac{\cos^2\varphi\,\mathrm{sen}\theta\cos\theta}{r^2\,\mathrm{sen}^2\varphi}\frac{\delta w}{\delta\theta}+\frac{\cos\varphi\,\mathrm{sen}\theta\cos\theta}{r^2\,\mathrm{sen}\varphi}\frac{\delta^2 w}{\delta\varphi\,\delta\theta}+$$

$$+\frac{\cos^2\theta}{r}\frac{\delta w}{\delta r}+\frac{\mathrm{sen}\theta\cos\theta}{r}\frac{\delta^2 w}{\delta\theta\,\delta r}+\frac{\cos\varphi\cos^2\theta}{r^2\,\mathrm{sen}\varphi}\frac{\delta w}{\delta\varphi}+$$

$$+\frac{\cos\varphi\,\mathrm{sen}\theta\cos\theta}{r^2\,\mathrm{sen}\varphi}\frac{\delta^2 w}{\delta\theta\,\delta\varphi}-\frac{\mathrm{sen}\theta\cos\theta}{r^2\,\mathrm{sen}^2\varphi}\frac{\delta w}{\delta\theta}+\frac{\cos^2\theta}{r^2\,\mathrm{sen}^2\varphi}\frac{\delta^2 w}{\delta\theta^2}$$

Y finalmente:

$$\frac{\delta w}{\delta z}=\frac{z}{r}\frac{\delta w}{\delta r}-\frac{\mathrm{sen}\varphi}{r}\frac{\delta w}{\delta\varphi}=\cos\varphi\frac{\delta w}{\delta r}-\frac{\mathrm{sen}\varphi}{r}\frac{\delta w}{\delta\varphi}=k$$

$$\frac{\delta^2 w}{\delta z^2}=\frac{\delta}{\delta z}\frac{\delta w}{\delta z}=\frac{\delta k}{\delta z}=\cos\varphi\frac{\delta k}{\delta r}-\frac{\mathrm{sen}\varphi}{r}\frac{\delta k}{\delta\varphi}=$$

$$=\cos\varphi\frac{\delta}{\delta r}\left(\cos\varphi\frac{\delta w}{\delta r}-\frac{\mathrm{sen}\varphi}{r}\frac{\delta w}{\delta\varphi}\right)-\frac{\mathrm{sen}\varphi}{r}\frac{\delta}{\delta\varphi}\left(\cos\varphi\frac{\delta w}{\delta r}-\frac{\mathrm{sen}\varphi}{r}\frac{\delta w}{\delta\varphi}\right)=$$

$$= \cos^2\varphi \frac{\delta^2 w}{\delta r^2} - \operatorname{sen}\varphi\cos\varphi\left(-\frac{1}{r^2}\frac{\delta w}{\delta\varphi} + \frac{1}{r}\frac{\delta^2 w}{\delta r\delta\varphi}\right) -$$

$$- \frac{\operatorname{sen}\varphi}{r}\left(-\operatorname{sen}\varphi\frac{\delta w}{\delta r} + \cos\varphi\frac{\delta^2 w}{\delta\varphi\delta r}\right) + \frac{\operatorname{sen}\varphi}{r^2}\left(\cos\varphi\frac{\delta w}{\delta\varphi} + \operatorname{sen}\varphi\frac{\delta^2 w}{\delta\varphi^2}\right) =$$

$$= \cos^2\varphi\frac{\delta^2 w}{\delta r^2} + \frac{\operatorname{sen}\varphi\cos\varphi}{r^2}\frac{\delta w}{\delta\varphi} - \frac{\operatorname{sen}\varphi\cos\varphi}{r}\frac{\delta^2 w}{\delta r\delta\varphi} +$$

$$+ \frac{\operatorname{sen}^2\varphi}{r}\frac{\delta w}{\delta r} - \frac{\operatorname{sen}\varphi\cos\varphi}{r}\frac{\delta^2 w}{\delta\varphi\delta r} + \frac{\operatorname{sen}\varphi\cos\varphi}{r^2}\frac{\delta w}{\delta\varphi} + \frac{\operatorname{sen}^2\varphi}{r^2}\frac{\delta^2 w}{\delta\varphi^2}$$

Para obtener el laplaciano de w, debemos sumar todos los términos correspondientes a las segundas derivadas de w respecto a x, y, z. Para ello, agruparemos primero los términos coeficientes comunes de cada derivada parcial, y realizaremos las simplificaciones que correspondan:

$\left.\frac{\delta^2 w}{\delta r^2}\right)$ $$\left(\operatorname{sen}^2\varphi\cos^2\theta + \operatorname{sen}^2\varphi\operatorname{sen}^2\theta + \cos^2\varphi\right)\frac{\delta^2 w}{\delta r^2} =$$

$$= \left[\operatorname{sen}^2\varphi\left(\cos^2\theta + \operatorname{sen}^2\theta\right) + \cos^2\varphi\right]\frac{\delta^2 w}{\delta r^2} = \left(\operatorname{sen}^2\varphi + \cos^2\varphi\right)\frac{\delta^2 w}{\delta r^2} = \frac{\delta^2 w}{\delta r^2}$$

$\left.\frac{\delta w}{\delta\varphi}\right)$ $$\left(-\frac{\operatorname{sen}\varphi\cos\varphi\cos^2\theta}{r^2} - \frac{\operatorname{sen}\varphi\cos\varphi\cos^2\theta}{r^2} + \frac{\cos\varphi\operatorname{sen}^2\theta}{r^2\operatorname{sen}\varphi} - \frac{\operatorname{sen}\varphi\cos\varphi\operatorname{sen}^2\theta}{r^2} -\right.$$

$$\left. - \frac{\operatorname{sen}\varphi\cos\varphi\operatorname{sen}^2\theta}{r^2} + \frac{\cos\varphi\cos^2\theta}{r^2\operatorname{sen}\varphi} + \frac{\operatorname{sen}\varphi\cos\varphi}{r^2} + \frac{\operatorname{sen}\varphi\cos\varphi}{r^2}\right)\frac{\delta w}{\delta\varphi} =$$

$$= -\frac{2\operatorname{sen}\varphi\cos\varphi\cos^2\theta}{r^2} - \frac{2\operatorname{sen}\varphi\cos\varphi\operatorname{sen}^2\theta}{r^2} +$$

$$\left. + \frac{2\operatorname{sen}\varphi\cos\varphi}{r^2} + \frac{\cos\varphi\operatorname{sen}^2\theta}{r^2\operatorname{sen}\varphi} + \frac{\cos\varphi\cos^2\theta}{r^2\operatorname{sen}\varphi}\right)\frac{\delta w}{\delta\varphi} =$$

$$= \left[-\frac{2\operatorname{sen}\varphi\cos\varphi\left(\cos^2\theta + \operatorname{sen}^2\theta\right)}{r^2} + \frac{2\operatorname{sen}\varphi\cos\varphi}{r^2} +\right.$$

$$\left. + \frac{\cos\varphi\left(\operatorname{sen}^2\theta + \cos^2\theta\right)}{r^2\operatorname{sen}\varphi}\right]\frac{\delta w}{\delta\varphi} = \frac{\cos\varphi}{r^2\operatorname{sen}\varphi}\frac{\delta w}{\delta\varphi}$$

$\left.\frac{\delta^2 w}{\delta r\delta\varphi}\right)$ $$\left(\frac{\operatorname{sen}\varphi\cos\varphi\cos^2\theta}{r} + \frac{\operatorname{sen}\varphi\cos\varphi\cos^2\theta}{r} + \frac{\operatorname{sen}\varphi\cos\varphi\operatorname{sen}^2\theta}{r} +\right.$$

$$\left. + \frac{\operatorname{sen}\varphi\cos\varphi\operatorname{sen}^2\theta}{r} - \frac{\operatorname{sen}\varphi\cos\varphi}{r} - \frac{\operatorname{sen}\varphi\cos\varphi}{r}\right)\frac{\delta^2 w}{\delta r\delta\varphi} =$$

$$= \left(\frac{2\operatorname{sen}\varphi\cos\varphi\cos^2\theta}{r} + \frac{2\operatorname{sen}\varphi\cos\varphi\operatorname{sen}^2\theta}{r} - \frac{2\operatorname{sen}\varphi\cos\varphi}{r}\right)\frac{\delta^2 w}{\delta r\delta\varphi} =$$

$$= \left(\frac{2\operatorname{sen}\varphi\cos\varphi\left(\cos^2\theta + \operatorname{sen}^2\theta\right)}{r} - \frac{2\operatorname{sen}\varphi\cos\varphi}{r}\right)\frac{\delta^2 w}{\delta r\delta\varphi} = 0$$

$$\frac{\delta w}{\delta \theta}\Big)\ \left(\frac{\operatorname{sen}\theta\cos\theta}{r^2}+\frac{\cos^2\varphi\operatorname{sen}\theta\cos\theta}{r^2\operatorname{sen}^2\varphi}+\frac{\operatorname{sen}\theta\cos\theta}{r^2\operatorname{sen}^2\varphi}-\right.$$

$$\left.-\frac{\operatorname{sen}\theta\cos\theta}{r^2}-\frac{\cos^2\varphi\operatorname{sen}\theta\cos\theta}{r^2\operatorname{sen}^2\varphi}-\frac{\operatorname{sen}\theta\cos\theta}{r^2\operatorname{sen}^2\varphi}\right)\frac{\delta w}{\delta\theta}=0$$

$$\frac{\delta^2 w}{\delta r\,\delta \theta}\Big)\ \left(-\frac{\operatorname{sen}\theta\cos\theta}{r}-\frac{\operatorname{sen}\theta\cos\theta}{r}+\frac{\operatorname{sen}\theta\cos\theta}{r}+\frac{\operatorname{sen}\theta\cos\theta}{r}\right)\frac{\delta^2 w}{\delta r\,\delta\theta}=0$$

$$\frac{\delta w}{\delta r}\Big)\ \left(\frac{\cos^2\theta\cos^2\theta}{r}+\frac{\operatorname{sen}^2\theta}{r}+\frac{\cos^2\varphi\operatorname{sen}^2\theta}{r}+\frac{\cos^2\theta}{r}+\frac{\operatorname{sen}^2\varphi}{r}\right)\frac{\delta w}{\delta r}=$$

$$=\left(\frac{\cos^2\varphi(\cos^2\theta+\operatorname{sen}^2\theta)}{r}+\frac{\operatorname{sen}^2\theta+\cos^2\theta}{r}+\frac{\operatorname{sen}^2\varphi}{r}\right)\frac{\delta w}{\delta r}=$$

$$=\left(\frac{\cos^2\varphi}{r}+\frac{1}{r}+\frac{\operatorname{sen}^2\varphi}{r}\right)\frac{\delta w}{\delta r}=\left(\frac{(\cos^2\varphi+\operatorname{sen}^2\varphi)}{r}+\frac{1}{r}\right)\frac{\delta w}{\delta r}=\frac{2}{r}\frac{\delta w}{\delta r}$$

$$\frac{\delta^2 w}{\delta \varphi^2}\Big)\ \left(\frac{\cos^2\varphi\cos^2\theta}{r^2}+\frac{\cos^2\varphi\operatorname{sen}^2\theta}{r^2}+\frac{\operatorname{sen}^2\varphi}{r^2}\right)\frac{\delta^2 w}{\delta\varphi^2}=$$

$$\left(\frac{\cos^2\varphi(\cos^2\theta+\operatorname{sen}^2\theta)}{r^2}+\frac{\operatorname{sen}^2\varphi}{r^2}\right)\frac{\delta^2 w}{\delta\varphi^2}=\frac{\cos^2\varphi+\operatorname{sen}^2\varphi}{r^2}\frac{\delta^2 w}{\delta\varphi^2}=\frac{1}{r^2}\frac{\delta^2 w}{\delta\varphi^2}$$

$$\frac{\delta^2 w}{\delta \varphi\,\delta\theta}\Big)\ \left(-\frac{\cos\varphi\operatorname{sen}\theta\cos\theta}{r^2\operatorname{sen}^2\varphi}-\frac{\cos\varphi\operatorname{sen}\theta\cos\theta}{r^2\operatorname{sen}^2\varphi}+\frac{\cos\varphi\operatorname{sen}\theta\cos\theta}{r^2\operatorname{sen}^2\varphi}+\frac{\cos\varphi\operatorname{sen}\theta\cos\theta}{r^2\operatorname{sen}^2\varphi}\right)\frac{\delta^2 w}{\delta\varphi\,\delta\theta}=0$$

$$\frac{\delta^2 w}{\delta \theta^2}\Big)\ \left(\frac{\operatorname{sen}^2\theta}{r^2\operatorname{sen}^2\varphi}+\frac{\cos^2\theta}{r^2\operatorname{sen}\varphi}\right)\frac{\delta^2 w}{\delta\theta^2}=\frac{\operatorname{sen}^2\theta+\cos^2\theta}{r^2\operatorname{sen}^2\varphi}\frac{\delta^2 w}{\delta\theta^2}=\frac{1}{r^2\operatorname{sen}^2\varphi}\frac{\delta^2 w}{\delta\theta^2}$$

Sumamos ahora los términos que quedan tras las simplificaciones:

$$\nabla^2 w=\frac{\delta^2 w}{\delta r^2}+\frac{\cos\varphi}{r^2\operatorname{sen}\varphi}\frac{\delta w}{\delta\varphi}+\frac{2}{r}\frac{\delta w}{\delta r}+\frac{1}{r^2}\frac{\delta^2 w}{\delta\varphi^2}+\frac{1}{r^2\operatorname{sen}^2\varphi}\frac{\delta^2 w}{\delta\theta^2}$$

Pero observemos que:

$$\frac{\delta}{\delta r}\left(r^2\frac{\delta w}{\delta r}\right)=2r\frac{\delta w}{\delta r}+r^2\frac{\delta^2 w}{r^2}$$

y dividiendo por r^2:

$$\frac{1}{r^2}\frac{\delta}{\delta r}\left(r^2\frac{\delta w}{\delta r}\right)=\frac{2}{r}\frac{\delta w}{\delta r}+\frac{\delta^2 w}{\delta r^2}$$

Asimismo:

$$\frac{\delta}{\delta\varphi}\left(\operatorname{sen}\varphi\frac{\delta w}{\delta\varphi}\right)=\cos\varphi\frac{\delta w}{\delta\varphi}+\operatorname{sen}\varphi\frac{\delta^2 w}{\delta\varphi^2}$$

y dividiendo por $r^2\operatorname{sen}\varphi$

$$\frac{1}{r^2\operatorname{sen}\varphi}\frac{\delta}{\delta\varphi}\left(\operatorname{sen}\varphi\frac{\delta w}{\delta\varphi}\right)=\frac{\cos\varphi}{r^2\operatorname{sen}\varphi}\frac{\delta w}{\delta\varphi}+\frac{1}{r^2}\frac{\delta^2 w}{\delta\varphi^2}$$

Sustituyendo, obtenemos finalmente la expresión del laplaciano del campo escalar w en coordenadas esféricas:

$$\boxed{\nabla^2 w=\frac{1}{r^2}\frac{\delta}{\delta r}\left(r^2\frac{\delta w}{\delta r}\right)+\frac{1}{r^2\operatorname{sen}\varphi}\frac{\delta}{\delta\varphi}\left(\operatorname{sen}\varphi\frac{\delta w}{\delta\varphi}\right)+\frac{1}{r^2\operatorname{sen}^2\varphi}\frac{\delta^2 w}{\delta\theta^2}}$$

Si en lugar de un campo escalar consideramos un campo vectorial $\vec{\Psi} = \Psi_x \vec{i} + \Psi_y \vec{j} + \Psi_z \vec{k}$, el laplaciano de $\vec{\Psi}$ se define como (Apostol, Calculus 2, p. 543) $\nabla^2 \vec{\Psi} = (\nabla^2 \Psi_x)\vec{i} + (\nabla^2 \Psi_y)\vec{j} + (\nabla^2 \Psi_z)\vec{k}$. Si aplicamos la expresión hallada del laplaciano en coordenadas esféricas a los campos escalares Ψ_x, Ψ_y, Ψ_z:

$$\nabla^2 \vec{\Psi} = \left[\frac{1}{r^2}\frac{\delta}{\delta r}\left(r^2 \frac{\delta \Psi_x}{\delta r}\right) + \frac{1}{r^2 \operatorname{sen}\varphi}\frac{\delta}{\delta \varphi}\left(\operatorname{sen}\varphi \frac{\delta \Psi_x}{\delta \varphi}\right) + \frac{1}{r^2 \operatorname{sen}^2\varphi}\frac{\delta^2 \Psi_x}{\delta \theta^2}\right]\vec{i} +$$

$$+ \left[\frac{1}{r^2}\frac{\delta}{\delta r}\left(r^2 \frac{\delta \Psi_y}{\delta r}\right) + \frac{1}{r^2 \operatorname{sen}\varphi}\frac{\delta}{\delta \varphi}\left(\operatorname{sen}\varphi \frac{\delta \Psi_y}{\delta \varphi}\right) + \frac{1}{r^2 \operatorname{sen}^2\varphi}\frac{\delta^2 \Psi_y}{\delta \theta^2}\right]\vec{j} +$$

$$+ \left[\frac{1}{r^2}\frac{\delta}{\delta r}\left(r^2 \frac{\delta \Psi_z}{\delta r}\right) + \frac{1}{r^2 \operatorname{sen}\varphi}\frac{\delta}{\delta \varphi}\left(\operatorname{sen}\varphi \frac{\delta \Psi_z}{\delta \varphi}\right) + \frac{1}{r^2 \operatorname{sen}^2\varphi}\frac{\delta^2 \Psi_z}{\delta \theta^2}\right]\vec{k} =$$

$$= \frac{1}{r^2}\left[\frac{\delta}{\delta r}\left(r^2 \frac{\delta \Psi_x}{\delta r}\right)\vec{i} + \frac{\delta}{\delta r}\left(r^2 \frac{\delta \Psi_y}{\delta r}\right)\vec{j} + \frac{\delta}{\delta r}\left(r^2 \frac{\Psi_z}{\delta r}\right)\vec{k}\right] +$$

$$+ \frac{1}{r^2 \operatorname{sen}\varphi}\left[\frac{\delta}{\delta \varphi}\left(\operatorname{sen}\varphi \frac{\delta \Psi_x}{\delta \varphi}\right)\vec{i} + \frac{\delta}{\delta \varphi}\left(\operatorname{sen}\varphi \frac{\delta \Psi_y}{\delta \varphi}\right)\vec{j} + \frac{\delta}{\delta \varphi}\left(\operatorname{sen}\varphi \frac{\delta \Psi_z}{\delta \varphi}\right)\vec{k}\right] +$$

$$+ \frac{1}{r^2 \operatorname{sen}^2\varphi}\left(\frac{\delta^2 \Psi_x}{\delta \theta^2}\vec{i} + \frac{\delta^2 \Psi_y}{\delta \theta^2}\vec{j} + \frac{\delta^2 \Psi_z}{\delta \theta^2}\vec{k}\right) =$$

$$= \frac{1}{r^2}\frac{\delta}{\delta r}\left(r^2 \frac{\delta \Psi_x}{\delta r}\vec{i} + r^2 \frac{\delta \Psi_y}{\delta r}\vec{j} + r^2 \frac{\delta \Psi_z}{\delta r}\right)\vec{k} +$$

$$+ \frac{1}{r^2 \operatorname{sen}\varphi}\frac{\delta}{\delta \varphi}\left(\operatorname{sen}\varphi \frac{\delta \Psi_x}{\delta \varphi}\vec{i} + \operatorname{sen}\varphi \frac{\delta \Psi_y}{\delta \varphi}\vec{j} + \operatorname{sen}\varphi \frac{\Psi_z}{\delta \varphi}\vec{k}\right) +$$

$$+ \frac{1}{r^2 \operatorname{sen}^2\varphi}\left(\frac{\delta^2 \Psi_x}{\delta \theta^2}\vec{i} + \frac{\delta^2 \Psi_y}{\delta \theta^2}\vec{j} + \frac{\delta^2 \Psi_z}{\delta \theta^2}\vec{k}\right) =$$

$$= \frac{1}{r^2}\frac{\delta}{\delta r}\left[r^2\left(\frac{\delta \Psi_x}{\delta r}\vec{i} + \frac{\delta \Psi_y}{\delta r}\vec{j} + \frac{\delta \Psi_z}{\delta r}\vec{k}\right)\right] +$$

$$+ \frac{1}{r^2 \operatorname{sen}\varphi}\frac{\delta}{\delta \varphi}\left[\operatorname{sen}\varphi\left(\frac{\delta \Psi_x}{\delta \varphi}\vec{i} + \frac{\delta \Psi_y}{\delta \varphi}\vec{j} + \frac{\delta \Psi_z}{\delta \varphi}\vec{k}\right)\right] +$$

$$+ \frac{1}{r^2 \operatorname{sen}^2\varphi}\left(\frac{\delta^2 \Psi_x}{\delta \theta^2}\vec{i} + \frac{\delta^2 \Psi_y}{\delta \theta^2}\vec{j} + \frac{\delta \Psi_z}{\delta \theta^2}\vec{k}\right)$$

Pero las derivadas parciales de un campo vectorial $\vec{\Psi}$ son las funciones vectoriales que tienen como componentes las derivadas parciales de las componentes de $\vec{\Psi}$ (Apostol, análisis matemático, p. 418; Apostol, Calculus, 2, p. 349; es decir:

$$\frac{\delta\vec{\Psi}}{\delta r} = \frac{\delta\Psi_x}{\delta r}\vec{i} + \frac{\delta\Psi_y}{\delta r}\vec{j} + \frac{\delta\Psi_z}{\delta r}\vec{k}$$

$$\frac{\delta\vec{\Psi}}{\delta\varphi} = \frac{\delta\Psi_x}{\delta\varphi}\vec{i} + \frac{\delta\Psi_y}{\delta\varphi}\vec{j} + \frac{\delta\Psi_z}{\delta\varphi}\vec{k}$$

$$\frac{\delta^2\vec{\Psi}}{\delta\theta^2} = \frac{\delta}{\delta\theta}\frac{\delta\vec{\Psi}}{\delta\theta} = \frac{\delta}{\delta\theta}\left(\frac{\delta\Psi_x}{\delta\theta}\vec{i} + \frac{\delta\Psi_y}{\delta\theta}\vec{j} + \frac{\delta\Psi_z}{\delta\theta}\vec{k}\right) =$$

$$= \frac{\delta^2\Psi_x}{\delta\theta^2}\vec{i} + \frac{\delta^2\Psi_y}{\delta\theta^2}\vec{j} + \frac{\delta^2\Psi_z}{\delta\theta^2}\vec{k}$$

Por tanto, sustituyendo:

$$\boxed{\nabla^2\vec{\Psi} = \frac{1}{r^2}\frac{\delta}{\delta r}\left(r^2\frac{\delta\vec{\Psi}}{\delta r}\right) + \frac{1}{r^2\operatorname{sen}\varphi}\frac{\delta}{\delta\varphi}\left(\operatorname{sen}\varphi\frac{\delta\vec{\Psi}}{\delta\varphi}\right) + \frac{1}{r^2\operatorname{sen}^2\varphi}\frac{\delta^2\vec{\Psi}}{\delta\theta^2}}$$

Luego la expresión hallada para el laplaciano en coordenadas esféricas es también válida para campos vectoriales.